Double algebraic inequality with geometric interpretation of its right hand side.
https://www.linkedin.com/feed/update/urn:li:activity:6708641381012828160
Let $x, y,$ and z be nonnegative real numbers such that $x^2+y^2+z^2+xyz=4$.
Prove that $0 \le xy+yz+zx-xyz \le 2$.
Solution by Arkady Alt, San Jose, California, USA.
Since by AM-GM inequality $4-xyz=x^2+y^2+z^2 \ge 3(x^2y^2z^2)^{1/3} \Leftrightarrow$
$3(xyz)^{2/3}+xyz-4 \le 0 \Leftrightarrow \left((xyz)^{1/3}-1\right)\left((xyz)^{1/3}+2\right)^2 \le 0 \Leftrightarrow xyz \le 1.$
and $xy+yz+zx \ge 3(x^2y^2z^2)^{1/3} \ge 3xyz$ then $xy+yz+zx-xyz \ge 2xyz \ge 0$.
Note that $xy+yz+zx-xyz \le 2$ holds if at least one of the x,y,z equal zero. Indeed, let $z=0$. Then $x^2+y^2=4$ implies $xy \le 2$ since $xy \le \dfrac{x^2+y^2}{2}$.
Thus, for further we can assume $x,y,z>0$.
Since all positive solutions of equation $x^2+y^2+z^2+xyz=4$ can be represented in the form $x=2\cos\alpha, y=2\cos\beta, z=2\cos\gamma$, where $\alpha,\beta,\gamma \in (0,\pi/2)$ and $\alpha+\beta+\gamma=\pi$ then inequality $xy+yz+zx-xyz \le 2$ becomes $4\sum\cos\alpha\cos\beta-8\cos\alpha\cos\beta\cos\gamma \le 2 \Leftrightarrow$
(**1**) $\quad 2\sum\cos\alpha\cos\beta-4\cos\alpha\cos\beta\cos\gamma \le 1.$
Let ABC be some acute triangle with angles α,β,γ and let R,r,s be,respectively, circumradius, inradius and semiperimeter of $\triangle ABC$.
Since $\cos\alpha\cos\beta\cos\gamma=\dfrac{s^2-(2R+r)^2}{4R^2}$ and $\sum\cos\alpha\cos\beta=\dfrac{s^2+r^2-4R^2}{2R^2}$
inequality (**1**) becomes

$$\frac{s^2+r^2-4R^2}{2R^2}-\frac{s^2-(2R+r)^2}{R^2} \le 1 \Leftrightarrow s^2+r^2-4R^2-2\left(s^2-(2R+r)^2\right) \le 2R^2$$

and we have $2R^2+2\left(s^2-(2R+r)^2\right)-(s^2+r^2-4R^2)=s^2-2R^2-8Rr-3r^2 \ge 0$,
because $2R^2+8Rr+3r^2 \le s^2$ (**Walker's Inequality for acute angled triangle**).
Thus, for $x,y,z>0$ such that $x^2+y^2+z^2+xyz=4$ inequality $xy+yz+zx-xyz \le 2$ can be considered as algebraic equivalent of **Walker's Inequality**.